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# GENERALIZED RUNGE - LENZ VECTOR IN TAUB -NUT SPINNING SPACE

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## Abstract

The generalized Killing equations and the symmetries of Taub-NUT spinning space are investigated. For spinless particles the Runge-Lenz vector defines a constant of motion directly, whereas for spinning particles it now requires a non-trivial contribution from spin. The generalized Runge-Lenz vector for spinning Taub-NUT space is completely evaluated.

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# 1 Introduction

Spinning particles, such as Dirac fermions, can be described by pseudo-classical mechanics models involving anti-commuting c-numbers for the spin-degrees of freedom. The configuration space of spinning particles (spinning space) is an extension of an ordinary Riemannian manifold, parametrized by local co-ordinates  $\{x^\mu\}$ , to a graded manifold parametrized by local co-ordinates  $\{x^\mu, \psi^\mu\}$ , with the first set of variables being Grassmann-even (commuting) and the second set Grassmann-odd (anti-commuting) [1-9].

In this paper we investigate the Runge-Lenz vector for the motion of pseudo-classical spinning point particles in the Euclidean Taub-NUT space [10]. The Kaluza-Klein monopole was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional theory, adding the time co-ordinate in a trivial way [11, 12]. Its line element is expressed as:

$$\begin{aligned} ds_5^2 &= -dt^2 + ds_4^2 \\ &= -dt^2 + V^{-1}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] \\ &\quad + V(r)[dx^5 + \vec{A}(\vec{r}) d\vec{r}]^2 \end{aligned} \quad (1)$$

where  $\vec{r}$  denotes a three-vector  $\vec{r} = (r, \theta, \varphi)$  and the gauge field  $\vec{A}$  is that of a monopole

$$\begin{aligned} A_r = A_\theta &= 0, \quad A_\varphi = 4m(1 - \cos \theta) \\ \vec{B} &= \text{rot} \vec{A} = \frac{4m\vec{r}}{r^3}. \end{aligned} \quad (2)$$

The function  $V(r)$  is

$$V(r) = \left(1 + \frac{4m}{r}\right)^{-1} \quad (3)$$

and the so called NUT singularity is absent if  $x^5$  is periodic with period  $16\pi m$  [13].

It is convenient to make the co-ordinate transformation

$$4m(\chi + \varphi) = -x^5 \quad (4)$$

with  $0 \leq \chi < 4\pi$ , which converts the four-dimensional line element  $ds_4$  into

$$\begin{aligned} ds_4^2 &= V^{-1}(r)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2] + 16m^2 V(r)[d\chi + \cos \theta d\varphi]^2 \\ &= g_{\mu\nu} dx^\mu dx^\nu. \end{aligned} \quad (5)$$

Remarkably, the same object has re-emerged in the study of monopole scattering. Slow Bogomolny-Prasad-Sommerfield monopoles move along geodesics in a four-dimensional curved space with the line element  $ds_4$  [14-16].

The geodesic motion of a spinless particle of unit mass in (5) can be derived from the action:

$$\mathcal{S} = \int_a^b d\tau \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu. \quad (6)$$

Here and in the following the overdot denotes an ordinary proper-time derivative  $d/d\tau$ .

The invariance of the metric (5) under spatial rotations and  $\chi$  translations is generated by four Killing vectors

$$D^{(\alpha)} = R^{(\alpha)\mu} \partial_\mu, \quad \alpha = 1, \dots, 4 \quad (7)$$

where

$$\begin{aligned} D^{(1)} &= -\sin \varphi \frac{\partial}{\partial \theta} - \cos \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\cos \varphi}{\sin \theta} \frac{\partial}{\partial \chi} \\ D^{(2)} &= \cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cot \theta \frac{\partial}{\partial \varphi} + \frac{\sin \varphi}{\sin \theta} \frac{\partial}{\partial \chi} \\ D^{(3)} &= \frac{\partial}{\partial \varphi} \\ D^{(4)} &= \frac{\partial}{\partial \chi}. \end{aligned} \quad (8)$$

In the purely bosonic case these invariances would correspond to conservation of angular momentum and "relative electric charge" [16-18]:

$$\vec{j} = \vec{r} \times \vec{p} + q \frac{\vec{r}}{r}. \quad (9)$$

$$q = 16m^2 V(r) (\dot{\chi} + \cos \theta \dot{\varphi}) \quad (10)$$

where

$$\vec{p} = \frac{1}{V(r)} \dot{\vec{r}} \quad (11)$$

is the "mechanical momentum" which is only part of the momentum canonically conjugate to  $\vec{r}$ . Energy, given by

$$E = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu = \frac{1}{2} V^{-1}(r) \left[ \dot{\vec{r}}^2 + \left( \frac{q}{4m} \right)^2 \right] \quad (12)$$

is also conserved,  $\Pi_\mu$  being the covariant momentum.

Finally, there is a conserved vector analogous to the Runge- Lenz vector of the Coulomb problem:

$$\begin{aligned} \vec{\mathcal{K}} &= \vec{K}_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \\ &= \vec{p} \times \vec{j} + \left( \frac{q^2}{4m} - 4mE \right) \frac{\vec{r}}{r}. \end{aligned} \quad (13)$$

Its existence is rather surprising in view of the complexity of motion in the Taub- NUT space [16-19].

These results, combined, imply that the trajectories are conic sections [16-18].

## 2 Motion in spinning space

The extension of the Euclidean Taub- NUT space with additional fermionic dimensions, parametrized by vectorial Grassmann co-ordinate  $\{\psi^\mu\}$  follows naturally. An action for the geodesics of spinning space is

$$\mathcal{S} = \int_a^b d\tau \left( \frac{1}{2} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu \frac{D\psi^\nu}{D\tau} \right) \quad (14)$$

where the covariant derivative of  $\psi^\mu$  is defined by

$$\frac{D\psi^\mu}{D\tau} = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma_{\lambda\nu}^\mu \psi^\nu. \quad (15)$$

The concept of Killing vector can be generalized to the case of spinning manifolds. For this purpose it is necessary to consider variations of  $x^\mu$  and  $\psi^\mu$  which leave the action  $\mathcal{S}$  invariant modulo boundary terms :

$$\delta\mathcal{S} = \int_a^b d\tau \frac{d}{d\tau} \left( \delta x^\mu p_\mu - \frac{i}{2} \delta\psi^\mu g_{\mu\nu} \psi^\nu - \mathcal{J}(x, \dot{x}, \psi) \right) \quad (16)$$

where  $p_\mu$  is the canonical momentum conjugate to  $x^\mu$

$$p_\mu = g_{\mu\nu} \dot{x}^\nu + \frac{i}{2} \Gamma_{\mu\nu;\lambda} \psi^\lambda \psi^\nu = \Pi_\mu + \frac{i}{2} \Gamma_{\mu\nu;\lambda} \psi^\lambda \psi^\nu. \quad (17)$$

From Noether's theorem, if the equations of motion are satisfied, the quantity  $\mathcal{J}$  is a constant of motion.

If we expand  $\mathcal{J}$  in a power series in the covariant momentum

$$\mathcal{J}(x, \dot{x}, \psi) = \mathcal{J}^{(0)}(x, \psi) + \sum_{n=1}^{\infty} \frac{1}{n!} \Pi^{\mu_1} \dots \Pi^{\mu_n} \mathcal{J}_{\mu_1 \dots \mu_n}^{(n)}(x, \psi) \quad (18)$$

then  $\mathcal{J}$  is a constant of motion if its components satisfy a generalization of the Killing equation [6, 7] :

$$\mathcal{J}_{(\mu_1 \dots \mu_n; \mu_{n+1})}^{(n)} + \frac{\partial \mathcal{J}_{(\mu_1 \dots \mu_n)}^{(n)}}{\partial \psi^\sigma} \Gamma_{\mu_{n+1}\lambda}^\sigma \psi^\lambda = \frac{i}{2} \psi^\sigma \psi^\lambda R_{\sigma\lambda\nu(\mu_{n+1}} \mathcal{J}_{\mu_1 \dots \mu_n)}^{(n+1)\nu}. \quad (19)$$

In general the symmetries of a spinning-particle model can be divided into two classes. First, there are four independent *generic* symmetries which exist in any theory [6,7] :

1. Proper-time translations and the corresponding constant of motion is the energy  $E$  (12)
2. Supersymmetry generated by the supercharge

$$Q = \Pi_\mu \psi^\mu \quad (20)$$

3. Chiral symmetry generated by the chiral charge

$$\Gamma_* = \frac{1}{4!} \sqrt{g} \epsilon_{\mu\nu\lambda\sigma} \psi^\mu \psi^\nu \psi^\lambda \psi^\sigma \quad (21)$$

4. Dual supersymmetry, generated by the dual supercharge

$$Q^* = \frac{1}{3!} \sqrt{g} \epsilon_{\mu\nu\lambda\sigma} \Pi^\mu \psi^\nu \psi^\lambda \psi^\sigma. \quad (22)$$

The second kind of conserved quantities, called *non-generic*, depend on the explicit form of the metric  $g_{\mu\nu}(x)$ . Let us examine in detail the generalized Killing Eq.(19) for the specific case of Taub- NUT space. For this purpose we shall write the power expansion (18) in the form

$$\mathcal{J}(x, \dot{x}, \psi) = B + R_\mu \Pi^\mu + \frac{1}{2!} K_{\mu\nu} \Pi^\mu \Pi^\nu + \frac{1}{3!} L_{\mu\nu\lambda} \Pi^\mu \Pi^\nu \Pi^\lambda + \dots \quad (23)$$

with some notations similar to the ones used in eqs.(7) and (13). The generalized Killing eq.(19) reduces for the lowest compoments to

$$B_{,\mu} + \frac{\partial B}{\partial \psi^\sigma} \Gamma_{\mu\kappa}^\sigma \psi^\kappa = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\kappa\mu} R^\kappa \quad (24)$$

$$R_{(\mu;\nu)} + \frac{\partial R_{(\mu}}{\partial \psi^\sigma} \Gamma_{\nu)\kappa}^\sigma \psi^\kappa = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\kappa(\mu} K_{\nu)}^\kappa \quad (25)$$

$$K_{(\mu\nu;\lambda)} + \frac{\partial K_{(\mu\nu}}{\partial \psi^\sigma} \Gamma_{\lambda)\kappa}^\sigma \psi^\kappa = \frac{i}{2} \psi^\rho \psi^\sigma R_{\rho\sigma\kappa(\mu} L_{\nu\lambda)}^\kappa. \quad (26)$$

The purely bosonic ( $\psi$  - independent) parts of these equations reduces to the Killing equations for the isometries of the usual Taub- NUT space discussed in the previous chapter. The first equation (24) implies that  $B$  is an irrelevant constant, while the second equation (25) is the standard equation for the Killing vectors  $R_\mu^{(\alpha)}$  ( $\alpha = 1, \dots, 4$ ). The third equation (26) will be analysed in the next section.

The first generalized Killing equation (24) shows that with each Killing vector  $R_\mu^{(\alpha)}$  there is an associated Killing scalar  $B^{(\alpha)}$ . This equation has been solved in Ref.[20] for the spinning Taub- NUT space using for the Killing vectors  $R_\mu^{(\alpha)}$  the expressions (7) and (8). We get the constants of motion for the spinning Taub- NUT space in the form:

$$\vec{J} = \vec{B} + \vec{j} \quad (27)$$

$$J^{(4)} = B^{(4)} + q \quad (28)$$

where  $\vec{J} = (J^{(1)}, J^{(2)}, J^{(3)})$  and  $\vec{B} = (B^{(1)}, B^{(2)}, B^{(3)})$ . Eq.(27) shows that the conserved total angular momentum is the sum of the angular momentum  $\vec{j}$  (9) and the spin angular momentum contribution described by  $\vec{B}$ . Similarly, from eq.(28), we conclude that the "relative electric charge"  $q$  (10) is no longer conserved and the new conserved quantity  $J^{(4)}$  received a spin dependent part.

### 3 Runge- Lenz vector in spinning Taub- NUT space

The Runge- Lenz vector  $\vec{K} = (K^{(1)}, K^{(2)}, K^{(3)})$  given in eq.(13) is quadratic in 4- velocities in the usual Taub- NUT space and the 3- Killing tensors [21]  $K_{\mu\nu}^{(\alpha)}$  ( $\alpha = 1, 2, 3$ ) satisfy a generalized Killing equation

$$K_{(\mu\nu;\lambda)}^{(\alpha)} = 0 \quad \alpha = 1, 2, 3 \quad (29)$$

which is exactly the purely bosonic part of eq.(26). Eq.(29) can be verified explicitly and it was analysed in detail in [17] where it is shown the origin of these extra-conserved quantities for the geodesic motions in the Taub- NUT space. The existence of these 3- Killing tensors (Stackel- Killing tensors) can be related to the existence on Taub- NUT space of a Killing- Yano 2- form  $f_{\mu\nu} = -f_{\nu\mu}$  such that [22]

$$K_{\nu}^{\mu} = f_{\lambda}^{\mu} f_{\nu}^{\lambda} \quad (30)$$

A similar situation is found in the Kerr- Newman solutions of the combined Einstein- Maxwell equations [23-28]. The extension of these results to the spinning Kerr- Newman space is done in [9].

In this section we shall evaluate the generalized Runge- Lenz vector for the Taub- NUT space. For this purpose we shall solve eq.(25) using in the r.h.s the Killing tensor components which appear in (13). The complete Killing vectors can be written in the following form:

$$\mathcal{R}_{\mu}^{(\alpha)} = R_{\mu}^{(\alpha)} + S_{\mu}^{(\alpha)} \quad , \alpha = 1, 2, 3 \quad (31)$$

where  $R_{\mu}^{(\alpha)}$  are known from the scalar case, eqs.(7),(8) and they correspond to the angular momentum (9). The  $\psi$  - dependent parts of the Killing vectors  $S_{\mu}^{(\alpha)}$  contribute to the Runge- Lenz vector for the spinning space

$$\vec{\mathcal{K}} = \vec{K}_{\mu\nu} \cdot \dot{x}^{\mu} \dot{x}^{\nu} + \vec{S}_{\mu} \cdot \dot{x}^{\mu} \quad (32)$$

The evaluation of the last term involves some long and tedious calculations. The results are the following:

$$\begin{aligned}
S_\mu^{(1)} \cdot \dot{x}^\mu = & \left[ -\frac{1}{2}(4m+r) \cos \theta \cos \varphi \cdot S^{r\theta} + \right. \\
& 2mr \cos \theta \sin \varphi \cdot S^{\theta\varphi} + 2mr \sin \varphi \cdot S^{\theta\chi} + \\
& \left. \frac{1}{2}(4m+r) \sin \theta \sin \varphi \cdot S^{r\varphi} + 2mr \sin \theta \cos \theta \cos \varphi \cdot S^{\varphi\chi} \right] \dot{r} + \\
& \left[ \frac{1}{2}r(4m+r) \sin \theta \cos \varphi \cdot S^{r\theta} - \frac{2mr(6m+r)}{4m+r} \cos \theta \sin \varphi \cdot S^{r\varphi} - \right. \\
& \frac{2mr(6m+r)}{4m+r} \sin \varphi \cdot S^{r\chi} + \\
& \left. \frac{1}{2}r^2(6m+r) \sin \theta \sin \varphi S^{\theta\varphi} - 2mr^2 \sin^2 \theta \cos \varphi S^{\varphi\chi} \right] \dot{\theta} + \\
& \left[ \frac{2mr(6m+r)}{4m+r} \cos \theta \sin \varphi S^{r\theta} + \right. \\
& \left( \frac{1}{2}r(4m+r) \sin^3 \theta \cos \varphi + \frac{128m^4r}{(4m+r)^3} \sin \theta \cos^2 \theta \cos \varphi \right) S^{r\varphi} - \\
& \left( 2mr + \frac{4m^2r}{4m+r} - \frac{128m^4r}{(4m+r)^3} \right) \sin \theta \cos \theta \cos \varphi S^{r\chi} + \\
& \left( \frac{r^2(32m^3 + 64m^2r + 14mr^2 + r^3)}{2(4m+r)^2} \sin^2 \theta \cos \theta \cos \varphi + \right. \\
& \left. \frac{16m^3r^2}{(4m+r)^2} \cos^3 \theta \cos \varphi \right) S^{\theta\varphi} + \\
& \left( 2mr^2 \sin^2 \theta \cos \varphi + \frac{16m^3r^2}{(4m+r)^2} \cos^2 \theta \cos \varphi \right) S^{\theta\chi} - \\
& \left. \frac{16m^3r^2}{(4m+r)^2} \sin \theta \cos \theta \sin \varphi S^{\varphi\chi} \right] \dot{\varphi} + \\
& \left[ \frac{2mr(6m+r)}{4m+r} \sin \varphi S^{r\theta} + \frac{128m^4r}{(4m+r)^3} \sin \theta \cos \varphi S^{r\chi} + \right. \\
& \left( 2mr + \frac{4m^2r}{4m+r} + \frac{128m^4r}{(4m+r)^3} \right) \sin \theta \cos \theta \cos \varphi S^{r\varphi} + \\
& \left( \frac{16m^3r^2}{(4m+r)^2} \cos^2 \theta \cos \varphi - \right. \\
& \left. \frac{64m^3r^2 + 16m^2r^3 + 2mr^4}{(4m+r)^2} \sin^2 \theta \cos \varphi \right) S^{\theta\varphi} + \\
& \left. \frac{16m^3r^2}{(4m+r)^2} \cos \theta \cos \varphi S^{\theta\chi} - \frac{16m^3r^2}{(4m+r)^2} \sin \theta \sin \varphi S^{\varphi\chi} \right] \dot{\chi} \quad (33)
\end{aligned}$$

$$\begin{aligned}
S_\mu^{(3)} \cdot \dot{x}^\mu = & \left[ \frac{1}{2}(4m+r) \sin \theta S^{r\theta} - 2mr \sin^2 \theta S^{\varphi\chi} \right] \dot{r} + \\
& \left[ \frac{1}{2}r(4m+r) \cos \theta S^{r\theta} - 2mr^2 \sin \theta \cos \theta S^{\varphi\chi} \right] \dot{\theta} + \\
& \left[ \left( \frac{128m^4r}{(4m+r)^3} \cos^3 \theta + \frac{1}{2}r(4m+r) \sin^2 \theta \cos \theta \right) S^{r\varphi} + \right. \\
& \left( \frac{128m^4r}{(4m+r)^3} \cos^2 \theta + \frac{2mr(6m+r)}{4m+r} \sin^2 \theta \right) S^{r\chi} - \\
& \left( \frac{1}{2}r^2(6m+r) \sin^3 \theta + \frac{48m^3r^2}{(4m+r)^2} \sin \theta \cos^2 \theta \right) S^{\theta\varphi} + \\
& \left. mr^2 \left( 2 - \frac{16m^2}{(4m+r)^2} \right) \sin \theta \cos \theta S^{\theta\chi} \right] \dot{\varphi} + \\
& \left[ \left( -2mr \sin^2 \theta - \frac{4m^2r}{4m+r} \sin^2 \theta + \frac{128m^4r}{(4m+r)^3} \cos^2 \theta \right) S^{r\varphi} + \right. \\
& \frac{128m^4r}{(4m+r)^3} \cos \theta S^{r\chi} - \frac{16m^3r^2}{(4m+r)^2} \sin \theta S^{\theta\chi} - \\
& \left. 2mr^2 \left( 1 + \frac{24m^2}{(4m+r)^2} \right) \sin \theta \cos \theta S^{\theta\varphi} \right] \dot{\chi}
\end{aligned} \tag{34}$$

The component  $S_\mu^{(2)} \cdot \dot{x}^\mu$  can be obtain from  $S_\mu^{(1)} \cdot \dot{x}^\mu$ , eq.(33), using the substitutions :

$$\begin{aligned}
\sin \varphi &\longrightarrow -\cos \varphi \\
\cos \varphi &\longrightarrow \sin \varphi
\end{aligned} \tag{35}$$

In eqs.(33),(34) we used the quantity

$$S^{\mu\nu} = -i\psi^\mu\psi^\nu \tag{36}$$

which can be regarded as the spin- polarization tensor of the particle [1-9].

## 4 Concluding remarks

The main purpose of this work has been the evaluation of the Runge- Lenz vector for the spinning Taub- NUT space. Combining this result with those from Ref.[20] we have a complete description of the motion of spinning particles in Taub- NUT space.

Unfortunately the formulae are quite intricate as compared to the scalar case ( such as eqs. (9), (13)). We mention that similar involved formulae



appear also in the study of the Schwarzschild [8] and Kerr- Newman [9] spinning spaces.

Extension of the analysis of the hidden symmetries of Taub- NUT space from Ref. [17] for the spinning case is possible and necessary. In general it is desirable to have a deeper understanding of the role of the Runge- Lenz vector for the motion of spinning particles. Having described the dynamical symmetries for the classical problem, the next step is to go into the quantum mechanical picture.

These problems are under investigations.

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